Groups Elementarily Equivalent to a Free 2-nilpotent Group of Finite Rank

Alexei G. Myasnikov   Mahmood Sohrabi

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Abstract

In this paper we find a characterization for groups elementarily equivalent to a free nilpotent group G of class 2 and arbitrary finite rank.

Contents

1 Introduction

In this paper we give an algebraic description of groups elementarily equivalent to a given free nilpotent class 2 group of an arbitrary finite rank.

1.1 Elementary classification problem in groups

Importance of the elementary (logical) classification of algebraic structures was emphasized in the works of A.Tarski and A.Malcev. In general, the problem of elementary classification requires to characterize, in somewhat algebraic terms, all algebraic structures (perhaps, from a given class) elementarily equivalent to a given one. Recall, that two algebraic structures $\mathcal{A}$ and $\mathcal{B}$ in a language L are elementarily equivalent ($\mathcal{A} \equiv \mathcal{B}$) if they have the same first-order theories in L (undistinguishable in the first order logic in L).

Tarski's results [40] on elementary theories of $\mathbb{R}$ and $\mathbb{C}$ (algebraically closed and real closed fields), as well as, the subsequent results of J. Ax and S. Kochen [1, 2, 3] and Yu. Ershov [14, 15, 16] on elementary theories of $\mathbb{Q}_p$ (p-adically closed fields) became an algebraic classic and can be found in many text-books on model theory.

One of the initial influential results on elementary classification of groups is due to W.Szmielew - she classified elementary theories of abelian groups in terms of "Szmielew's" invariants [39] (see also [13, 22, 4]). For non-abelian groups, the main inspiration, perhaps, was the famous Tarski's problem whether free non-abelian groups of finite rank are elementarily equivalent or not. This problem has been open for many years and only recently was solved in affirmative in [18, 38]. In contrast, free solvable (or nilpotent) groups of finite rank are elementarily equivalent if and only if they are isomorphic (A. Malcev [19]). Indeed, in these cases the abelianization $G/[G,G]$ of the group G (hence the rank of G) is definable (interpretable) in G by first-order formulas, hence the result.

In [21] A. Malcev described elementary equivalence among classical linear groups. Namely, he showed that if $\mathcal{G} \in \{GL, PGL, SL, PSL\}$, $n, m \geq 3$, K and F are fields of characteristic zero, then $\mathcal{G}(F)_m \equiv \mathcal{G}(K)_n$ if and only if $m = n$ and $F \equiv K$. It turned out later that this type of results can be obtained via ultrapowers by means of the theory of abstract isomorphisms of such groups. In

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this approach one argues that if the groups $\mathcal{G}(F)_m$ and $\mathcal{G}(K)_n$ are elementarily equivalent then their ultrapowers over a non-principal ultrafilter ω are isomorphic. Since these ultrapowers are again groups of the type $\mathcal{G}(F^*)_m$ and $\mathcal{G}(K^*)_n$ (where F^* and K^* are the corresponding ultrapowers of the fields) the result follows from the description of abstract isomorphisms of such groups (which are semi-algebraic, so they preserve the algebraic scheme and the field). It follows that the results, similar to the ones mentioned above, hold for many algebraic and linear groups. We refer to [36] and [8, 9] for details. On the other hand, many "geometric" properties of algebraic groups are just first-order definable invariants of these groups, viewed as abstract groups (no geometry, only multiplication). For example, the geometry of a simple algebraic group is entirely determined by its group multiplication (see [41, 35, 34]), which readily implies the celebrated Borel-Tits theorem on abstract isomorphisms of simple algebraic groups.

Other large classes of groups where the elementary classification problem is relatively well-understood are the classes of finitely generated nilpotent groups and algebraic nilpotent groups. This is the main subject of the paper and we discuss it below. We finish a short survey on elementary equivalence of groups in a result related to the solutions of the Tarski's problem. It is known now that a finitely generated group G is elementarily equivalent to a free non-abelian group if and only if G is a regular NTQ-group ([18]), which are the same as ω-residually free towers ([38]). These results generated a new waive of research on elementary classification in classes of groups which are very different from solvable or algebraic groups, namely, in hyperbolic groups, relatively hyperbolic groups, free products of groups, right angled Artin groups, etc. On the other hand, research on large scale geometry of soluble groups, in particular, their quasi-isometric invariants, comes suspiciously close to the study of their first-order invariants. That gives a new impetus to learn more on elementarily equivalence in the class of solvable groups.

1.2   On elementary classification of nilpotent groups

There are several principal results known on elementary theories of nilpotent groups. In his pioneering paper [20] A. Malcev showed that a ring R with unit can be defined by first-order formulas in the group $UT_3(R)$ of unitriangular matrices over R (viewed as an abstract group). In particular, the ring of integers $\mathbb{Z}$ is definable in the group $UT_3(\mathbb{Z})$, which is a free 2-nilpotent group of rank 2. In [17] Yu. Ershov proved that the group $UT_3(\mathbb{Z})$ (hence the ring $\mathbb{Z}$) is definable in any finitely generated nilpotent group G, which is not virtually abelian. It follows immediately that the elementary theory of G is undecidable. On the elementary classification side the main research was on M.Kargapolov's conjecture: two finitely generated nilpotent groups are elementarily equivalent if and only if they are isomorphic. In [42] B. Zilber gave a counterexample to the Kargapolov's conjecture. In the break-through papers [23, 24, 25] A. Myasnikov and V. Remeslennikov proved that the Kargapolov's conjecture holds "essentially" true in the class of nilpotent $\mathbb{Q}$-groups (i.e., divisible torsion-free

nilpotent groups) finitely generated as $\mathbb{Q}$-groups. Indeed, it turned out that two such groups G and H are elementarily equivalent if their *cores* $\bar{G}$ and $\bar{H}$ are isomorphic and G and H either simultaneously coincide with their cores or they do not. Here the core of G is uniquely defined as a subgroup $\bar{G} \leq G$ such that $Z(\bar{G}) \leq [\bar{G}, \bar{G}]$ and $G = \bar{G} \times G_0$, for some abelian $\mathbb{Q}$-group G_0. Developing this approach further A.Myasnikov described in [28, 29] all groups elementarily equivalent to a given finitely generated nilpotent K-group G over an arbitrary field of characteristic zero. Here by a K-group we understand P. Hall nilpotent K-powered groups, which are the same as K-points of nilpotent algebraic groups, or unipotent K-groups. Again, the crucial point here is that the geometric structure of the group G (including the fields of definitions of the components of G and their related structural constants) are first-order definable in G, viewed as an abstract group. Furthermore, these ideas shed some light on the Kargapolov's conjecture - it followed that two finitely generated elementarily equivalent nilpotent groups G and H are isomorphic, provided one of them is a *core* group. In this case G is a *core* group if $Z(G) \leq I([G, G])$, where $I([G, G])$ is the isolator of the commutant $[G, G]$. Finally, F.Oger showed in [32] that two finitely generated nilpotent groups G and H are elementarily equivalent if and only if they are essentially isomorphic, i.e., $G \times \mathbb{Z} \simeq H \times \mathbb{Z}$. However, the full classification problems for finitely generated nilpotent groups is currently wide open. In a series of papers [5, 6, 7] O. Belegradek completely characterized groups which are elementarily equivalent to a nilpotent group $UT_n(\mathbb{Z})$ for $n \geq 3$. It is easy to see that (via ultrapowers) that if $\mathbb{Z} \equiv R$ for some ring R then $UT_n(\mathbb{Z}) = UT_n(R)$. However, it has been shown in [6, 7] that there are groups elementarily equivalent to $UT_n(\mathbb{Z})$ which are not isomorphic to any group of the type $UT_n(R)$ as above (quasi-unitriangular groups).

1.3   Results and the structure of the paper

In this paper we generalize O. Belegradek's results from [5] on characterizing groups elementary equivalent to the group $UT_3(\mathbb{Z})$, which is, as mentioned above, a free nilpotent group of class 2 and rank 2. We start with describing a certain class $N_{2,n}$ of groups such that $G = N_{2,n}(\mathbb{Z})$ is a free 2-nilpotent group of rank n for a natural number $n \geq 2$. The class $N_{2,2}$ coincides with the class of upper unitriangular groups over a commutative associative unitary rings R. We introduce various definitions for a "basis" of an $N_{2,n}$ group over R and show that these bases are first order definable. As should be expected from the work of O. Belegradek on UT_3 groups the classes $N_{2,n}$ are not elementarily closed. To close the class we introduce a new type of groups $QN_{2,n}$, and prove that they give the elementary closure of the class of $N_{2,n}$ groups over commutative unitary rings. One of the most crucial steps in this work is "recovering" the ring of integers $\mathbb{Z}$ from G and showing that it is absolutely interpretable in G. To do so we use the method of bilinear mappings, due to A. Myasnikov [27, 30], in contrast with O. Belegradek who uses Mal'cev original idea from [20].

Now we describe contents of the sections. In Section 2 we present the basic facts and definitions from the theory of nilpotent groups, the theory of group

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extensions, model theory and model theory of bilinear mappings. The classes of free 2-nilpotent groups $N_{2,n}$ of rank n over commutative associative rings R with unit and some related concepts are discussed in Section 3. In Section 4 we introduce a new class of nilpotent groups $QN_{2,n}$, termed quasi $N_{2,n}$ groups. We give a characterization theorem for the $QN_{2,n}$ groups over R Section 6. This section contains the main result, Theorem 6.9, of this paper in which we prove that a group elementarily equivalent to a free 2-nilpotent group of arbitrary rank n is a group of the type $QN_{2,n}$ over a ring R with $R \equiv \mathbb{Z}$. In Section 7 we prove that the necessary condition above is also a sufficient condition, thus providing a complete characterization of groups elementarily equivalent to a given free 2-nilpotent groups of finite rank. In Section 8 we show an example of a group $QN_{2,n}$, which is elementarily equivalent to $N_{2,n}(\mathbb{Z})$ but is not an $N_{2,n}$ group over any commutative associative unitary ring.

2 Preliminaries and notation

2.1 Nilpotent groups

Let G be a group with a series of subgroups:

$$G = G_1 \geq G_2 \geq \ldots G_n \geq G_{n+1} = 1,$$

where each G_{i+1} is a normal subgroup of G and each factor G_i/G_{i+1} is an abelian group. Let G act on each factor G_i/G_{i+1} by conjugation, i.e.

$$g.xG_{i+1} =_{df} g^{-1}xgG_{i+1}.$$

If the above action of G on all the factors is trivial then the above series is called a *central series* and any group G with such a series is called a *nilpotent* group.

For elements x and y of a group G let $[x, y] = x^{-1}y^{-1}xy$. $[x, y]$ is called the *commutator* of the elements x and y. The subgroup $[G, G]$ is the subgroup of G generated by all $[x, y]$, $x, y \in G$. In general for H and K subgroups of G, $[H, K]$ is the subgroup of G generated by commutators $[x, y]$, $x \in H$ and $y \in K$. Let us define a series $\Gamma_1(G), \Gamma_2(G), \ldots$ of subgroups of G by setting

$$G = \Gamma_1(G), \quad \Gamma_{n+1}(G) = [\Gamma_n(G), G] \quad \text{for all } n > 1.$$

It can be easily checked that the above series is a central series. If c is the least number that $\Gamma_{c+1}(G) = 0$ then G is said to be a nilpotent group of *class c* or simply a *c-nilpotent* group. We call the series above the *lower central series* of the group G.

Let $Z(G)$ denote the center of a group G. We define a series of subgroups $Z_i(G)$ of G by setting

$$Z_1(G) = Z(G), \quad Z_{i+1}(G) = \{x \in G : xZ_i \in Z(G/Z_i(G))\}, \quad i > 1.$$

This series is also a central series and called the *upper central series* of the group G. If $Z_{n+1}(G) = G$ for some finite number n and c is the least such number then G is provably a c-nilpotent group.

Let $F(n)$ be the free group on n generators. Let G be a group isomorphic to the factor group $F(n)/\Gamma_{c+1}(F(n))$. Then G is called a *free nilpotent group of class c and rank n*, or equivalently a *free c-nilpotent group of rank n*.

For definitions and details regarding Mal'cev (canonical) basis of a finitely generated torsion free nilpotent group, groups admitting exponents in a binomial domain R or R-group for short and R-completions of finitely generated torsion free nilpotent groups we refer to [11].

2.2 Central and abelian extensions

Let A and B be abelian groups. Consider the short exact sequence:

$$0 \to A \xrightarrow{\mu} E \xrightarrow{\nu} B \to 0.$$

The group E is called an *abelian extension* of A by B if E is an abelian group. E is said to be a *central extension* of A by B if $\mu(A)$ sits inside the center of E, i.e. the action defined above is trivial. Obviously every abelian extension is central. It can be easily seen that every central extension of two abelian groups is a 2-nilpotent group. An extension:

$$0 \to A \xrightarrow{\mu'} E' \xrightarrow{\nu'} B \to 0$$

is *equivalent* to the extension above if there is an isomorphism $\eta : E \to E'$ such that $\nu' \circ \eta = \nu$ and $\eta \circ \mu = \mu'$. The relation "equivalence" defines an equivalence relation on the set of all central extensions of the abelian groups A and B.

We now review the relation between equivalence classes of central extensions of an abelian group A by an abelian group B and the group called the *second cohomology* group, $H^2(B, A)$.

A function $f : B \times B \to A$ is called a *2-cocycle* if

- $f(0, x) = f(x, 0) = 0$ for every x in B,

- $f(x + y, z) + f(x, y) = f(x, y + z) + f(y, z)$, for x, y and z in B.

By $B^2(B, A)$ we denote the set of all 2-cocycles $f : B \times B \to A$. A We can make the set $B^2(B, A)$ into an abelian group by letting addition of the corresponding functions be the point-wise addition. An element $f \in B^2(B, A)$ is called a *2-coboundary* if there exists a function $\psi : B \to A$ such that

$$f(x, y) = \psi(x + y) - \psi(x) - \psi(y).$$

We note the set of all 2-coboundaries by $Z^2(B, A)$. The group operation defined above on $B^2(B, A)$ makes $Z^2(B, A)$ into a subgroup. An element $f \in B^2(B, A)$ is called a *symmetric 2-cocycle* if

$$f(x, y) = f(y, x), \quad \forall x, y \in B.$$

By $S^2(B, A)$ we mean the subgroup of $B^2(B, A)$ consisting of all symmetric 2-cocycles. Set

$$H^2(B, A) =_{df} B^2(B, A)/Z^2(B, A),$$

and

$$Ext(B, A) =_{df} S^2(B, A)/(Z^2(B, A) \cap S^2(B, A)).$$

There is a 1-1 correspondence between elements of $H^2(B, A)$ and equivalence classes of central extensions of A by B. The same correspondence also exists between elements of $Ext(B, A)$ and equivalences classes of abelian extensions of A by B. We briefly explain one direction the correspondence. For more details we refer to [37]. let $f : B \times B \to A$ be a 2-cocycle. Define a group $E(f)$ by $E(f) = B \times A$ as sets with the multiplication

$$(b_1, a_1)(b_2, a_2) = (b_1 + b_2, a_1 + a_2 + f(b_1, b_2)) \quad a_1, a_2 \in A, \quad b_1, b_2 \in B.$$

The above operation is a group operation and the resulting extension is central. If f is symmetric then $E(f)$ is abelian. Moreover it can be verified that if $f, f' : B^($B, A)$ and $f - f' \in Z^2(B, A)$ then the extensions $E(f)$ and $E(f')$ are equivalent.

2.3   Structures, signatures and interpretations

A group G is considered to be the structure $\langle |G|, ., ^{-1}, 1 \rangle$ where $.$, $^{-1}$ and 1, name multiplication, inverse operation and the trivial element of the group respectively. We consider this signature as the *signature of groups*. We use $[x, y]$ as an abbreviation for $x^{-1}.y^{-1}.x.y$.

By an *algebraic structure* we mean a structure including functions only, constants aside. Strangely enough here we assume that algebraic structures consist only of predicates in addition to constant symbols. But in a sense what we mean is clear. Algebraic operations are considered as relations rather than functions.

Let $\mathfrak{U}$ be a structure and $\phi(x_1, \ldots, x_n)$ be a first order formula of the signature of $\mathfrak{U}$ with $x_1, \ldots, x_n$ free variables. Let $(a_1, \ldots, a_n) \in |\mathfrak{U}|^n$. We denote such a tuple by $\bar{a}$. The notation $\mathfrak{U} \models \phi(\bar{a})$ is intended to mean that the tuple $\bar{a}$ satisfies $\phi(\bar{x})$ when $\bar{x}$ is an abbreviation for the tuple $(x_1, \ldots, x_n)$ of variables. For definitions of a formula of a signature, free variables and satisfaction the reader should refer to [12].

Given a structure $\mathfrak{U}$ and a first order formula $\phi(x_1, \ldots, x_n)$ of the signature of $\mathfrak{U}$, $\phi(\mathfrak{U}^n)$ refers to $\{\bar{a} \in |\mathfrak{U}|^n : \mathfrak{U} \models \phi(\bar{a})\}$. Such a relation or set is called *first order definable without parameters*. If $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a first order formula of the signature of $\mathfrak{U}$ and $\bar{b}$ an m-tuple of elements of $\mathfrak{U}$ then $\psi(\mathfrak{U}^n, \bar{b})$ means $\{\bar{a} \in |\mathfrak{U}|^n : \mathfrak{U} \models \psi(\bar{a}, \bar{b})\}$. A set or relation like this is said to be *first order definable with parameters*.

Let $\mathfrak{U}$ be a structure of signature Σ. The *theory* $Th(\mathfrak{U})$ of the structure $\mathfrak{U}$ is the set:

$$\{\phi : \mathfrak{U} \models \phi, \phi \text{ a first order sentence of signature } \Sigma\}.$$

Finally two structures $\mathfrak{U}$ and $\mathfrak{B}$ of the signature Σ are *elementarily equivalent* if $Th(\mathfrak{U}) = Th(\mathfrak{B})$.

Let $\mathfrak{B}$ and $\mathfrak{U}$ be algebraic structures of signatures Δ and Σ respectively not having function symbols. In the following $\bar{x}$ refers to an n-tuple of variables

$(x_1, \ldots, x_n)$ and for each i, $\overline{y^i}$ refers to an m-tuple of variables $(y_1^i, \ldots, y_m^i)$. The structure $\mathfrak{U}$ is said to be *interpretable* in $\mathfrak{B}$ with parameters $\bar{b} \in |\mathfrak{B}|^n$ or *relatively interpretable* in $\mathfrak{B}$ if there is a set of first order formulas

$$\Psi = \{A(\bar{x}, \bar{y}), E(\bar{x}, \overline{y^1}, \overline{y^2}), \psi_\sigma(\bar{x}, \overline{y^1}, \ldots, \overline{y^{t_\sigma}}) : \sigma \text{ a predicate of signature } \Sigma\}$$

of the signature Δ such that

1. $A(\bar{b}) = \{\bar{a} \in |\mathfrak{B}|^m : \mathfrak{B} \models A(\bar{b}, \bar{a})\}$ is not empty,

2. $E(\bar{x}, \overline{y^1}, \overline{y^2})$ defines an equivalence relation $\epsilon_{\bar{b}}$ on $A(\bar{b})$,

3. if the equivalent class of a tuple of elements $\bar{a}$ from $A(\bar{b})$ modulo the equivalence relation $\epsilon_{\bar{b}}$ is denoted by $[\bar{a}]$, for every predicate σ of signature Σ, predicate P_σ is defined on $A(\bar{b})/\epsilon_{\bar{b}}$ by

$$P_\sigma([\bar{b}], [\overline{a^1}], \ldots [\overline{a^{t_\sigma}}]) \Leftrightarrow_{df} \mathfrak{B} \models \psi_\sigma(\bar{b}, \overline{a^1}, \ldots \overline{a^{t_\sigma}}),$$

4. the structures $\mathfrak{U}$ and $\Psi(\mathfrak{B}, \bar{b}) = \langle A(\bar{b})/\epsilon_{\bar{b}}, P_\sigma : \sigma \in \Sigma \rangle$ are isomorphic.

Let $\phi(x_1, \ldots, x_n)$ be a first order formula of the signature Δ and $\bar{b} \in \phi(\mathfrak{B}^n)$ be as above. If $\mathfrak{U}$ is interpretable in $\mathfrak{B}$ with the parameters $\bar{b}$ and $\mathfrak{B} \models \phi(\bar{b})$ then $\mathfrak{U}$ is said to be *regularly interpretable* in $\mathfrak{B}$ with the help of formula ϕ. If the tuple $\bar{b}$ is empty, $\mathfrak{U}$ is said be *absolutely interpretable* in $\mathfrak{B}$.

2.4 Bilinear mappings as model theoretic objects

All the results in this subsection are due to A. G. Myasnikov, [27] and [30].

Let M and N be exact R-modules for some commutative ring R. An R-module M is *exact* if $rm = 0$ for $r \in R$ and $0 \neq m \in M$ imply $r = 0$. Let's recall that an R-bilinear mapping $f : M \times M \to N$ is called *non-degenerate* in both variables if $f(x, M) = 0$ or $f(M, x) = 0$ implies $x = 0$. We call the bilinear map f, *"onto"* if N is generated by $f(x, y)$, $x, y \in M$. We associate two many sorted structures to every bilinear mapping described above. One of them

$$\mathfrak{U}_R(f) = \langle R, M, N, \delta, s_M, s_N \rangle,$$

where the predicate δ describes the mapping and s_M and s_N describe the actions of R on the modules M and N respectively. The other one,

$$\mathfrak{U}(f) = \langle M, N, \delta \rangle,$$

contains only a predicate δ describing the mapping f. It can be easily seen that the structure $\mathfrak{U}(f)$ is absolutely interpretable in $\mathfrak{U}_R(f)$. We intend to show that there is a ring $P(f)$ such that $\mathfrak{U}_{P(f)}(f)$ is absolutely interpretable in $\mathfrak{U}(f)$. Moreover this ring is the maximal ring relative to which f remains bilinear.

2.4.1 Enrichments of bilinear mappings

In this subsection are the modules are considered to be exact. Let M be an R-module and let $\mu : R \to P$ be an inclusion of rings. Then the P-module M is an P-*enrichment* of the R-module M with respect to μ if for every $r \in R$ and $m \in M$, $rm = \mu(r)m$. Let us denote the set of all R endomorphisms of the R-module M by $End_R(M)$. Suppose the R-module M admits a P-enrichment with respect to the inclusion of rings $\mu : R \to P$. Then every $\alpha \in P$ induces an R-endomorphism, $\phi_\alpha : M \to M$ of modules defined by $\phi_\alpha(m) = \alpha m$ for $m \in M$. This in turn induces an injection $\phi_P : P \to End_R(M)$ of rings. Thus we associate a subring of the ring $End_R(M)$ to every ring P with respect to which there is an enrichment of the R-module M.

Definition 2.1. *Let $f : M \times M \to N$ be an R-bilinear "onto" mapping and $\mu : R \to P$ be an inclusion of rings. The mapping f admits P-enrichment with respect to μ if the R-modules M and N admit P enrichments with respect to μ and f remains bilinear with respect to P. We denote such an enrichment by $E(f, P)$.*

We define an ordering $\leq$ on the set of enrichments of f by letting $E(f, P_1) \leq E(f, P_2)$ if and only if f as an P_1 bilinear mapping admits a P_2 enrichment with respect to inclusion of rings $P_1 \to P_2$. The largest enrichment $E_H(f, P(f))$ is defined in the obvious way. We shall prove existence of such an enrichment for a large class bilinear mappings.

Proposition 2.2. *If $f : M \times M \to N$ is a non-degenerate "onto" R-bilinear mapping over a commutative ring R, f admits the largest enrichment.*

2.4.2 Interpretability of the $P(f)$ structure

Let $f : M \times M \to N$ be a non-degenerate "onto" R-bilinear mapping for some commutative ring R. The mapping f is said to have *finite width* if there is a natural number S such that for every $u \in N$ there are x_i and y_i in M we have

$$u = \sum_{i=1}^{n} f(x_i, y_i).$$

The least such number, $w(f)$, is the *width* of f.

A set $E = \{e_1, \ldots e_n\}$ is a complete system for f if $f(x, E) = f(E, x) = 0$ for $x \in M$ implies $x = 0$. The cardinality of a complete system with minimal cardinality is denoted by $c(f)$.

Type of a bilinear mapping f, denoted by $\tau(f)$, is the pair $(w(f), c(f))$. The mapping f is said to be of finite type if $c(f)$ and $w(f)$ are both finite numbers.

Now we state the main theorem of this subsection:

Theorem 2.3. *Let $f : M \times M \to$ be non-degenerate "onto" bilinear mapping of finite type. Then the structure $\mathfrak{U}_{P(f)}(f)$ is absolutely interpretable in $\mathfrak{U}(f)$*

For proofs and details we refer the reader to [27] and [30] although we shall describe the ring $P(f)$ later.

3 $N_{2,n}$ groups

3.1 Definition of $N_{2,n}$ groups

Let R be a ring with unit and for an arbitrary natural number $n \geq 2$ consider the set of all $n + \binom{n}{2}$-tuples $((\alpha_i)_{1 \leq i \leq n}, (\gamma_{ij})_{1 \leq i < j \leq n})$ of elements of R where (i, j) are ordered lexicographically and the same order is assumed on the γ_{ij}. By $((-), (-))$ is meant a concatenation of two tuples. We denote this set by X. We drop the subscripts and denote the tuple only by $((\alpha_i), (\gamma_{ij}))$. Always $(\bar{0})$ means that all the coordinates are 0. Define a multiplication on this set by:

$$((\alpha_i), (\gamma_{ij}))((\beta_i), (\gamma'_{ij})) =_{df} ((\alpha_i + \beta_i), (\gamma_{ij} + \gamma'_{ij} + \alpha_i \beta_j)), \quad \alpha_i, \beta_i, \gamma_{ij}, \gamma'_{ij} \in R. \tag{1}$$

By $N_{2,n}(R)$ we mean the set X together with the operation above.

Lemma 3.1. *The operation defined in* (1) *makes $N_{2,n}(R)$ into a group.*

Proof. Let $x = ((\alpha_i), (\gamma_{ij}))$, $y = ((\beta_i), (\gamma'_{ij}))$ and $z = ((\delta_i), (\gamma''_{ij}))$ be elements of $N_{2,n}(R)$. Then

$$
\begin{aligned}
(xy)z &= ((\alpha_i + \beta_i), (\gamma_{ij} + \gamma'_{ij} + \alpha_i \beta_j))z \\
&= (((\alpha_i + \beta_i) + \delta_i), (((\gamma_{ij} + \gamma'_{ij}) + \gamma''_{ij} + \alpha_i \beta_j + (\alpha_i + \beta_i)\delta_j)) \\
&= ((\alpha_i + (\beta_i + \delta_i)), ((\gamma_{ij} + (\gamma'_{ij}) + \gamma''_{ij}) + \alpha_i(\beta_j + \delta_j) + \beta_i \delta_j) \\
&= ((\alpha_i), (\gamma_{ij}))((\beta_i + \delta_i), (\gamma'_{ij} + \gamma''_{ij} + \beta_i \delta_j)) \\
&= x(yz),
\end{aligned}
$$

which proves the associativity of the operation. The identity element is clearly $((\bar{0}), (\bar{0}))$ and if x is as above then $x^{-1} = ((-a_i), (a_i a_j - d_{ij}))$. So $N_{2,n}(R)$ is a group. $\square$

An isomorphic copy of $N_{2,n}(R)$ is called an $N_{2,n}$ *group over* R. If $\mathcal{R}$ is a class of rings, $N_{2,n}(\mathcal{R})$ is the class of all groups G such $G \cong N_{2,n}(R)$ for some ring R in $\mathcal{R}$. If $\mathcal{R}$ is the class of all rings a member of the class $N_{2,n}(\mathcal{R})$ is called an $N_{2,n}$ group. We note that $N_{2,2}(R) \cong UT_3(R)$.

Next proposition shows our main interest in $N_{2,n}$ groups. We postpone the proof to the end of Subsection 3.4.

Proposition 3.2. *If $\mathbb{Z}$ is the ring of integers then $N_{2,n}(\mathbb{Z})$ is a free 2-nilpotent group of rank n.*

3.2 Commutator subgroup and center of a $N_{2,n}$ group

Let G be a $N_{2,n}$ group over some ring R with unit. An elementary computation shows if $x = ((\alpha_i), (\gamma_{ij}))$ and $y = ((\beta_i), (\gamma'_{ij}))$ then we have

$$[x, y] = ((\bar{0}), (\alpha_i \beta_j - \beta_i \alpha_j)).$$

Now we can study the relation between the commutator subgroup $[G, G]$ and the center $Z(G)$ of G.

Lemma 3.3. *Let G be a $N_{2,n}$ group. Then $Z(G) = [G, G]$.*

Proof. By the equation for commutators obtained above it is clear that $[G, G]$ is the set of elements of the form $x = ((\bar{0}), (\gamma_{ij}))$, $\gamma_{ij} \in R$, $1 \leq i < j \leq n$. It is clear that for such x, $[x, y] = 1$ for every $y \in G$. So $[G, G] \subseteq Z(G)$. For the converse let $x = ((\alpha_i), (\gamma_{ij})) \in Z(G)$. If $y = ((\beta_i), (\gamma'_{ij}))$ is an arbitrary element of G then we must have $[x, y] = ((\bar{0}), (\alpha_i \beta_j - \beta_i \alpha_j)) = 1 = ((\bar{0}), (\bar{0}))$. Since this equality holds for all elements β_i and β_j of R it also holds if $\beta_j = 1$ and $\beta_i = 0$, for each $1 \leq i < j \leq n$. So all $\alpha_i = 0$, $1 \leq i \leq n - 1$. Setting $\beta_{n-1} = 1$ and $\beta_n = 0$ will obtain that $\alpha_n = 0$. So $x \in [G, G]$. $\square$

We note that as a consequence of Lemma 3.3 a $N_{2,n}$ group is 2-nilpotent.

3.3 Standard basis for a $N_{2,n}$ group

When all coordinates of an element $x = ((\alpha_i), (\gamma_{ij}))$ of $N_{2,n}(R)$ are zero except possibly the i-th coordinate then x is denoted by $g_i^{\alpha_i}$. If every coordinate of x is zero except possibly the ij-th coordinate x is denoted by $g_{ij}^{\gamma_{ij}}$. In particular $g_i^0 = g_{ij}^0 = 1$. We also assume that $g_i^1 = g_i$ for all $1 \leq i \leq n$ and $g_{ij}^1 = g_{ij}$ for all $1 \leq i < j \leq n$. By what has been shown above:

$$[g_i^\alpha, g_j^\beta] = g_{ij}^{\alpha\beta}, \quad \alpha, \beta \in R,$$

and $[g_i, g_j] = g_{ij}$. So $[g_i^\alpha, g_j^\beta] = g_{ij}^{\alpha\beta} = [g_i, g_j]^{\alpha\beta}$. Thus given an element $x = ((\alpha_i), (\gamma_{ij})) \in N_{2,n}(R)$ it is clear that

$$x = g_n^{\alpha_n} g_{n-1}^{\alpha_{n-1}} \ldots g_1^{\alpha_1} [g_1, g_2]^{\gamma_{12}} \ldots [g_1, g_n]^{\gamma_{1n}} [g_2, g_3]^{\gamma_{23}} \ldots [g_{n-1}, g_n]^{\gamma_{n-1,n}}. \quad (2)$$

or equivalently

$$x = g_n^{\alpha_n} g_{n-1}^{\alpha_{n-1}} \ldots g_1^{\alpha_1} g_{12}^{\delta_{12}} \ldots g_{1n}^{\delta_{1n}} g_{23}^{\delta_{23}} \ldots g_{n-1,n}^{\delta_{n-1,n}}. \quad (3)$$

Thus the set $\{g_i^\alpha | 1 \leq i \leq n, \alpha \in R\}$ is a generating set for $N_{2,n}(R)$. Moreover it should be clear that every element x of $N_{2,n}(R)$ has a unique representation of the form given in the equations (3) and (2). We call the elements $g_1, \ldots, g_n$ a *standard basis* for the group $N_{2,n}(R)$.

3.4 Centralizers of elements of the standard basis of a $N_{2,n}$ group

Consider a ring R with unit and a $N_{2,n}$ group G over R. Let $C_G(x)$ denote the centralizer of an element x of G in G. Now Set:

$$G_i =_{df} C_G(g_i), \quad 1 \leq i \leq n,$$

where $g_1, \ldots, g_n$ constitute the standard basis for G. Let $g_i^R = \{g_i^\alpha : \alpha \in R\}$. We first prove that $G_i = g_i^R \oplus Z(G)$.

Lemma 3.4. *For each $1 \leq i \leq n$, $G_i = g_i^R \oplus Z(G)$.*

Proof. Firstly we observe that g_i^R is a subgroup. Let $x = ((\alpha_i), (\gamma_{ij}))$ be an arbitrary element of G. By the discussion in Subsection 3.3, $x = g_n \ldots g_1 v$ when v is an element of the center $Z(G)$. Then

$$\begin{aligned}
[g_i, x] &= [g_i, g_n^{\alpha_n} \ldots g_1^{a_1} v] \\
&= [g_i, g_n^{\alpha_n} \ldots g^{\alpha_1}][g_i, v] \\
&= [g_i, g_n^{\alpha_n} \ldots g_1^{\alpha_1}] \\
&= g_{1i}^{-\alpha_i} \ldots g_{i-1,i}^{-\alpha_{i-1,i}} g_{i,i+1}^{\alpha_{i,i+1}} \ldots g_{jn}^{\alpha_n}
\end{aligned}$$

For x to be in G_i it is necessary that $[g_i, x] = 1$. So by the above equality $\alpha_j = 0$ for $i \neq j$. So x must have the form $g_i^a v$ for some a in R and v in $Z(G)$. It is also clear that $g_i^R \cap Z(G) = 1$. □

Corollary 3.5. *Each G_i, $1 \leq i \leq n$ is abelian.*

Proof. Clear by Lemma 3.4. □

Next define subgroups G_{ij} of G by

$$G_{ij} =_{df} [G_i, G_j], \quad 1 \leq i < j \leq n. \tag{4}$$

Lemma 3.6. *The equalities:*

$$G_{ij} = [G_i, g_j] = [g_i, G_j], \quad 1 \leq i < j \leq n,$$

hold, when G_{ij} are defined in Equation (4).

Proof. We shall prove $[g_i, G_j] = [G_i, G_j]$ for each $1 \leq i < j \leq n$. The direction $\subseteq$ is obvious. For the converse let $x \in G_i$ and $y \in G_j$. By Lemma 3.4, $x = g_i^a v$ and $y = g_j^b v'$ for some $a, b \in R$ and $v, v' \in Z(G)$. Thus,

$$\begin{aligned}
[g_i^\alpha v, g_j^\beta v'] &= [g_i^\alpha, g_j^\beta] \\
&= [g_i, g_j]^{\alpha\beta} \\
&= [g_i, g_j^{\alpha\beta}] \in [g_i, G_j].
\end{aligned}$$

The other equality can be proved similarly. □

The next thing we can verify is that

$$G_i \cap G_j = Z(G), \quad 1 \leq i, j \leq n. \tag{5}$$

Lemma 3.7. *Equations (5) hold in for the subgroups G_i, G_j and $Z(G)$ for each $1 \leq i, j \leq n$.*

Proof. Let the element x of G be such that $x \in G_i$ and $y \in G_j$. By Lemma 3.4, $x = g_i^\alpha v = g_j^\beta v'$ for some $\alpha, \beta \in R$ and $v, v' \in Z(G)$, implying that $\alpha = \beta = 0$. Therefore $x \in Z(G)$. The other direction is clear. □

We assemble the lemmas and corollaries above in a single proposition.

Proposition 3.8. *Let G be a $N_{2,n}$ group over a ring R with unit. Suppose $g_1,\ldots,g_n$ constitute the standard basis for G. Let $G_i = C_G(g_i)$, $1 \leq i \leq n$, and $G_{ij} = [G_i, G_j]$, $1 \leq i < j \leq n$. Then the following statements hold,*

1. *$G_{ij} = [G_i, g_j] = [g_i, G_j]$, $\quad 1 \leq i < j \leq n$,*

2. *$G_i \cap G_j = Z(G)$, $\quad 1 \leq i < j \leq n$,*

3. *$[G_i, G_i] = 1$, $\quad 1 \leq i \leq n$,*

4. *$[G, G] = Z(G)$,*

5. *every element of G can be written as $u_n u_{n-1} \ldots u_1 v$ where each $u_i \in G_i$ and $v \in Z(G)$ and each u_i is unique modulo the center. Moreover each $v \in Z(G)$ can be uniquely written as $u_{12} \ldots u_{1n} u_{23} \ldots u_{n-1,n}$ when $u_{ij} \in G_{ij}$.*

Proof. See Subsection 3.3, Lemmas 3.6, 3.7 and Corollary 3.5. $\qquad\qquad$ □

Now we are in a good set up to prove Proposition 3.2.

Proof. (**Proof of Proposition 3.2**) Let $F(n)$ be the free group on generators $u_1, \ldots, u_n$. Let $\Gamma_3(F(n))$ denote the third term of the lower central series of $F(n)$. Let $g_1, \ldots, g_n$ be elements of the standard basis for $N_{2,n}(\mathbb{Z})$. Note that $\{g_1, \ldots, g_n\}$ is a generating set for $N_{2,n}(\mathbb{Z})$. The mapping:

$$F(n)/\Gamma_3(F(n)) \longrightarrow N_{2,n}(\mathbb{Z}), \quad u_i \Gamma_3(F(n)) \mapsto g_i, \quad 1 \leq i \leq n,$$

is a well defined homomorphism of groups since $\Gamma_3(F(n))$ is generated by the simple commutators $[[u_i, u_j], u_k]$, $i \neq j$, and $[[g_i, g_j], g_k] = 1$ holds in $N_{2,n}(\mathbb{Z})$. It is also a surjection since $g_1, \ldots, g_n$ generate $N_{2,n}(\mathbb{Z})$.

We prove that it is also an injection. Notice that for every integer k and m,

$$[u_i^m \Gamma_3(F(n)), u_j^k \Gamma_3(F(n))] = [u_i, u_j]^{mk} \Gamma_3(F(n)) \tag{6}$$

holds for each pair u_i, u_j of elements in $\{u_1, \ldots, u_n\}$. So every element u in $F(n)/\Gamma_3(F(n))$ can be brought to the form in (2), g_i substituted by u_i for each $1 \leq i \leq n$. This can be done using the so-called collection process, applying the relations (6) and using the fact that all the commutators in $F(n)/\Gamma_3(F(n))$ belong to the center of the group. For example if u_i and u_j are such that $i < j$ and m and k are integers then:

$$u_i^m u_j^k \Gamma_3(F(n)) = u_i^k u_j^m [u_i^m, u_j^k] \Gamma_3(F(n))$$
$$= u_j^k u_i^m [u_i, u_j]^{mk} \Gamma_3(F(n)).$$

By repeating this process finitely many times and moving the commutators to the right hand side we arrive at the indicated form for any element of $F(n)/\Gamma_3(F(n))$. Therefore under the mapping defined above u gets mapped

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to an element g of $N_{2,n}(\mathbb{Z})$ with a representation exactly like what appears in (2). This form is unique so the element g is trivial in $N_{2,n}(\mathbb{Z})$ if and only if all the exponents in (2) are zero if and only if u is trivial in $F(n)/\Gamma_3(F(n))$. And we are done. $\qquad\square$

Remark 1. Assume $\{g_1, \ldots, g_n\}$ is the standard basis for $G = N_{2,n}(\mathbb{Z})$. Then it is not too hard to verify that the tuple $(g_n, g_{n-1}, \ldots, g_1, g_{12}, \ldots, g_{n,n-1})$ is a Mal'cev basis for G. So if R is a binomial domain by definition $N_{2,n}(R)$ is the Mal'cev R-completion of G an hence an R-group. We'll use this fact in the last section.

4 $QN_{n,2}$ groups

4.1 Definition of $QN_{n,2}$ groups

Let $f \in S^2(R^+, \oplus_{i=1}^{\binom{n}{2}} R^+$ be a symmetric 2-cocycle, where R is a ring with unit. Such a 2-cocycle has $\binom{n}{2}$ coordinates $f_{jk} : R^+ \times R^+ \longrightarrow R$, $1 \le j < k \le n$, each of which is a symmetric 2-cocycle. Now for each i, $1 \le i \le n$, let $f^i : S^2(R^+, \oplus_{i=1}^{\binom{n}{2}} R^+$ be a symmetric 2-cocycle with components f_{jk}^i, $1 \le j < k \le n$.

We define a new multiplication $\odot$ on the underlying set X of $N_{2,n}(R)$ by

$$((\alpha_i), (\gamma_{ij})) \odot ((\beta_i), (\gamma'_{ij})) = ((\alpha_i + \beta_i), (\gamma_{ij} + \gamma'_{ij} + \alpha_i \beta_j + \sum_{k=1}^{n} f_{ij}^k(\alpha_k, \beta_k))). \quad (7)$$

Lemma 4.1. *The set X is a group with respect to the multiplication $\odot$ defined in* (7).

Proof. Let $g^i : \oplus_{i=1}^n R^+ \times \oplus_{i=1}^n R^+ \to \oplus_{i=1}^{\binom{n}{2}}$ be defined by

$$g^i((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)) = f^i(\alpha_i, \beta_i).$$

It is easy to verify that $g^i \in B^2(\oplus_{i=1}^n R^+, \oplus_{i=1}^{\binom{n}{2}})$. Now it is clear that the $(X, \odot)$ is a central extension of $\oplus_{i=1}^n R^+$ by $\oplus_{i=1}^{\binom{n}{2}} R^+$ via a 2-cocycle $f = (f_{ij})_{1 \le i < j \le n}$ defined by

$$f_{ij}((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)) = \alpha_{ij} + \sum_{i=1}^{n} g_{ij}^i((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)).$$

$\qquad\square$

We denote the new group by $N_{2,n}(R, f^1 \ldots f^n)$. If $\mathcal{R}$ is a class of rings with unit, by $QN_{2,n}(\mathcal{R})$ we mean the class of all groups G such that $G \cong N_{2,n}(R, f^1, \ldots, f_n)$ for some ring R in $\mathcal{R}$ and symmetric 2-cocycles $f^i : R^+ \times R^+ \to R^{\binom{n}{2}}$, $i = 1, \ldots, n$. Such a group G is called a $QN_{2,n}$ group over R. If $\mathcal{R}$ is the class of all rings a member of the class $QN_{2,n}(\mathcal{R})$ is called a $QN_{2,n}$ group.

4.2 Commutator subgroup and center of a $QN_{2,n}$ group

Let G be a $QN_{2,n}$ group over a ring R with unit. To give a formula for the commutator of two elements we need to verify a basic fact about symmetric 2-cocycles.

Let $x = ((\alpha_i), (\gamma_{ij})$ and $y = ((\beta_i), (\gamma'_{ij}))$ be in G then

$$
\begin{aligned}
x^{(-1)} \odot y^{(-1)} \odot x \odot y = ((\bar{0}), (\gamma_{ij} + \gamma'_{ij} - \gamma_{ij} - \gamma'_{ij} \\
+ \alpha_i\alpha_j + \beta_i\beta_j + \alpha_i\beta_j + \alpha_i\beta_j + (-\alpha_i - \beta_i)(\alpha_j + \beta_j) \\
- \sum_{k=i}^{n} (f_{ij}^k(\alpha_k, -\alpha_k) - f_{ij}^k(\beta_k, -\beta_k) \\
+ f_{ij}^k(\alpha_k, \beta_k) + f_{ij}^k(-\alpha_k - \beta_k, \alpha_k + \beta_k)) \\
= ((\bar{0}), (\alpha_i\beta_j - \beta_i\alpha_j))
\end{aligned}
$$

by the above lemma. Thus commutators in $QN_{2,n}$ and $N_{2,n}$ groups coincide. So we have the lemma:

Lemma 4.2. *In a $QN_{2,n}$ group G, $Z(G) = [G, G]$.*

Proof. The proof goes through exactly like that of Lemma 3.3. $\qquad\square$

4.3 Standard basis for a $QN_{2,n}$ group

Again as in the $N_{2,n}$ groups we denote an element $((\alpha_i), (\gamma_{ij}))$ which has zeros everywhere except possibly at the i-th position by $g_i^{a_i}$ and the one which has zeros everywhere except possibly at ij-th position by $g_{ij}^{\gamma_{ij}}$. We call the set $\{g_1, \ldots, g_n\}$ the *standard basis* of the group $QN_{2,n}(R)$. Let us note that for a $QN_{2,n}$ group G over a ring R with unit and the standard basis $\{g_1, \ldots, g_n\}$, the quotient $G/Z(G)$ is a free module over R of rank n generated by $\{g_1 Z(G), \ldots, g_n Z(G)\}$. Moreover $Z(G) = [G, G]$ is a free R-module of rank $\binom{n}{2}$ generated by the $g_{ij} = [g_i, g_j]$, $1 \leq i < j \leq n$.

Proposition 4.3. *Let G be a $QN_{2,n}$ group over a ring R with unit, G_i for each $1 \leq i \leq n$ and G_{ij} for each $1 \leq i < j \leq n$ be defined as in proposition 3.8. Then all the conditions (1)-(5) in proposition 3.8 are also true in the group G.*

Proof. Similar to the proof of Proposition 3.8. $\qquad\square$

4.4 Generators and relations for a $QN_{2,n}$ group

Here we specify a set of generators and relations for a $QN_{2,n}$ group.

Lemma 4.4. *The group $G = N_{2,n}(R, f^1, \ldots, f^n)$ is generated by*

$$
\{g_i^a, g_{kl}^b : 1 \leq i \leq n, 1 \leq k < l \leq n, \quad \alpha, \beta \in R\},
$$

and defined by the relations:
(a) $[g_i^\alpha, g_j^\beta] = g_{ij}^{\alpha\beta}$, for all $1 \leq i < j \leq n$, $\alpha, \beta \in R$

(b) $[g_i^\alpha, g_{kl}^\beta] = [g_{rs}^\alpha, g_{kl}^\beta] = 1,$ for all $1 \le i \le n$, $1 \le k < l \le n$, $1 \le r < s \le n$ and $\alpha, \beta \in R$,

(c) $g_i^\alpha \odot g_i^\beta = g_i^{(\alpha+\beta)} g_{12}^{f_{12}^i(\alpha,\beta)} \ldots g_{n-1,n}^{f_{n-1,n}^i(\alpha,\beta)},$ for all $1 \le i \le n$, $\alpha, \beta \in R$

(d) $g_{ij}^\alpha \odot g_{ij}^\beta = g_{ij}^{\alpha+\beta},$ for all $1 \le i < j \le n$, $\alpha, \beta \in R$.

Proof. clearly the set

$$\mathcal{G} = \{g_i^a, g_{kl}^b | 1 \le i \le n, 1 \le k < l \le n, \quad \alpha, \beta \in R\},$$

is a generating set for G. Let F be the free group generated by the set $\mathcal{G}$ and $\mathcal{R}$ be the normal subgroup of F generated by the relations (a)-(d) above, multiplication $\odot$ taken to be concatenation. Now consider the group $\langle \mathcal{G} | \mathcal{R} \rangle$, the quotient of F by $\mathcal{R}$. Consider the mapping:

$$\langle \mathcal{G} | \mathcal{R} \rangle \longrightarrow G, \quad g_i^\alpha \mapsto g_i^\alpha, \quad g_{kl}^\alpha \mapsto g_{kl}^\alpha$$

for every $\alpha \in R$, $1 \le i \le n$ and $1 \le k < l \le n$. The map is a well-defined homomorphism since all the relations (a)-(d) hold in G. Every word W in $\langle \mathcal{G} | \mathcal{R} \rangle$ is equivalent to a word with the form given in (3), multiplication taken to be concatenation. this element gets mapped to an element g with the same form in the group G, multiplication taken to be $\odot$. This form is unique by Proposition 4.3. So g is trivial in G if and only if W is trivial in $\langle \mathcal{G} | \mathcal{R} \rangle$. The proposition is proved. □

5 $QN_{2,n}$ groups over commutative rings

Let G be a nilpotent group of class 2. We associate to G a bilinear map

$$f_G: \ G/[G,G] \times G/[G,G] \longrightarrow [G,G], \quad (x[G,G], y[G,G]) \mapsto [x,y]$$

$x, y \ inG$. Note that in a $QN_{2,n}$ group G, $[G,G] = Z(G)$. So the bilinear map f_G becomes

$$f_G: G/Z(G) \times G/Z(G), \quad (xZ(G), yZ(G)) \mapsto [x,y]$$

$x, y \in G$.

When G is a $QN_{2,n}$ the bilinear map f_G is both "onto" and non-degenerate so as in Section 2.4we can associate to it a commutative associative ring $P(f_G)$ with unit relative to which f_G is bilinear and $P(f_G)$ is the maximal such ring. Actually $P(f_G)$ is the set of all pairs

$$(\phi_1, \phi_0) \in E = End(G/Z(G)) \times End(Z(G)),$$

where $End(-)$ refers to the endomorphism ring of the corresponding group, which satisfy the identity

$$f_G(\phi_1(xZ(G)), yZ(G)) = f_G(xZ(G), \phi_1(yZ(G))) = \phi_0(f_G(x,y))$$

for all $x, y \in G$.

Lemma 5.1. *Let G be a $QN_{2,n}$ group over a commutative associative ring R with unit. Let $(\phi_1, \phi_0) \in P(f_G)$ and $x, y \in G$. Then $\phi_1(xZ(G)) = (xZ(G))^\gamma$ and $\phi_0(f_G(x,y)) = f_G(x,y)^\gamma$ for some $\gamma \in R$.*

Proof. Let $\{g_1, g_2, \ldots, g_n\}$ be the standard basis for G and $g_{ij} = [g_i, g_j]$ for $1 \leq i < j \leq n$. Since $g_i Z(G)$, $1 \leq i \leq n$, generate $G/Z(G)$ and g_{ij}, $1 \leq i < j \leq n$ generate $Z(G)$ it is enough to study the action of the ϕ_i on powers of basis elements.

Fix $1 \leq i \leq n$. We assume that $\phi_1(g_i Z(G)) = g_1^{\alpha_1} g_2^{\alpha_2} \ldots g_n^{\alpha_n} Z(G)$ for some $\alpha_k \in R$, $1 \leq k \leq n$. Now

$$
\begin{aligned}
g_{1j}^{\alpha_1} \cdots g_{j-1,j}^{\alpha_{j-1}} g_{j,j+1}^{-\alpha_{j+1}} \cdots g_{jn}^{-\alpha_n} &= [g_1^{\alpha_1} \cdots g_n^{\alpha_n}, g_i] \\
&= f_G(\phi_1(g_i Z(G)), g_i Z(G)) \\
&= \phi_0([g_i, g_i]) \\
&= 1
\end{aligned}
$$

So $\alpha_k = 0$, $k \neq i$. Set $\alpha_k = \gamma$. Now pick $1 \leq j \leq n$ such that $i \neq j$. By an argument like the one above we may conclude that there exists $\delta \in R$ such that $\phi_1(g_j Z(G)) = (g_j Z(G))^\delta$. Then

$$
\begin{aligned}
g_{ij}^\gamma &= f_G(\phi_1(g_i), g_j) \\
&= f_G(g_i, \phi_1(g_j)) \\
&= g_{ij}^\delta
\end{aligned}
$$

So $\gamma = \delta$. This proves that for any $1 \leq k \leq n$, $\phi_1(g_k Z(G)) = (g_k Z(G))^\gamma$.

Next let $\alpha \in R$ and pick $1 \leq i < j \leq n$. Suppose

$$
\phi_1(g_i^\alpha Z(G)) = g_1^{\alpha_1} \ldots g_n^{\alpha_n} Z(G).
$$

Then,

$$
\begin{aligned}
g_{1j}^{\alpha_1} \cdots g_{j-1,j}^{\alpha_{j-1}} g_{j,j+1}^{-\alpha_{j+1}} \cdots g_{jn}^{-\alpha_n} &= [g_1^{\alpha_1} \cdots g_n^{\alpha_n}, g_j] \\
&= f_G(\phi_1(g_i^\alpha Z(G)), g_j Z(G)) \\
&= f_G((g_i Z(G))^\alpha, \phi_1(g_j Z(G))) \\
&= [g_i^\alpha, g_j^\gamma] = g_{ij}^{\alpha\gamma}
\end{aligned}
$$

So $\alpha_i = \alpha\gamma$ and $\alpha_k = 0$ if $k \neq i, j$. To prove $\alpha_j = 0$ it is enough to consider $f_G(\phi_1((g_i Z(G))^\alpha), g_i Z(G)) = 1$. It is also clear that $\phi_0(g_{ij}^\alpha) = g_{ij}^{\alpha\gamma}$ for all $1 \leq i < j \leq n$.

Since ϕ_1 and ϕ_0 are endomorphisms the statement is clear now. $\square$

Proposition 5.2. *Let R be a commutative associative ring with unit and G be a $QN_{2,n}$ group over R. Then $P(f_G) \cong R$*

Proof. Define a mapping

$$
\eta : P(f_G) \to R, \quad (\phi_1, \phi_0) \mapsto \gamma_\phi
$$

where $\phi_1(xZ(G)) = (xZ(G))^{\gamma_\phi}$ for $x \in G$ and $\phi_0(y) = y^{\gamma_\phi}$ for $y \in Z(G)$. Such a γ_ϕ exists by Lemma 5.1. The mapping is well defined since if $(xZ(G))^{\gamma_1} = (xZ(G))^{\gamma_2}$ for all $x \in G$ then also $(g_iZ(G))^{\gamma_1} = (g_iZ(G))^{\gamma_2}$ which implies $\gamma_1 = \gamma_2$. Let γ be an element of the ring R. Define a triple (ϕ_1, ϕ_1, ϕ_0) where $\phi_1 \in End(G/Z(G))$ and $\phi_0 \in End(Z(G))$ by setting $\phi_1(xZ(G)) = (xZ(G))^\gamma$, for $x \in G$ and $\phi_0(z) = z^\gamma$, for $z \in Z(G)$. We show that $(\phi_1, \phi_1, \phi_0) \in P(f_G)$. Let $\{g_1, \ldots, g_n\}$ be the standard basis for G, $g_{ij} = [g_i, g_j]$, $1 \le i < j \le n$, $xZ(G) = (g_1 Z(G))^{\alpha_1} \cdots (g_n Z(G))^{\alpha_n}$ and $yZ(G) = (g_1 Z(G))^{\beta_1} \cdots (g_n Z(G))^{\beta_n}$ for $x, y \in G$. Then by associativity and commutativity of R,

$$
\begin{aligned}
f_G(x^\gamma Z(G), yZ(G)) &= f_G(g_1^{\alpha_1} \ldots g_n^{\alpha_n} Z(G), g_1^{\beta_1} \ldots g_n^{\beta_n} Z(G)) \\
&= g_{12}^{((\alpha_1\gamma)\beta_2 - \beta_1(\alpha_2\gamma))} \cdots g_{n-1,n}^{((\alpha_{n-1}\gamma)\beta_n - \beta_{n-1}(\alpha_n\gamma))} \\
[x,y]^\gamma &= (g_{12}^{(\alpha_1\beta_2 - \beta_1\alpha_2)} \cdots g_{n-1,n}^{(\alpha_{n-1}\beta_n - \beta_{n-1}\alpha_n)})^\gamma \\
&= g_{12}^{(\alpha_1(\beta_2\gamma) - \beta_1(\gamma\alpha_2))} \cdots g_{n-1,n}^{(\alpha_{n-1}(\beta_n\gamma) - (\beta_{n-1}\gamma)\alpha_n)} \\
&= f_G(xZ(G), y^\gamma Z(G)).
\end{aligned}
$$

So $(\phi_1, \phi_0) \in P(f_G)$. This proves the surjectivity of η. If $(\phi_1, \phi_0) \in P(f_G)$ maps to the zero of the ring R under the mapping η, it means that ϕ_1 and ϕ_0 are zero endomorphisms. Hence (ϕ_1, ϕ_0) is the zero of $P(f_G)$. Hence the mapping η is injective.

To prove that η is an additive homomorphism note that

$$
\eta((\phi_1, \phi_0) + (\psi_1, \psi_0)) = \eta((\phi_1 + \psi_1, \phi_0 + \psi_0))
$$

is an element γ of R such that for every $x \in G$,

$$
(xZ(G))^\gamma = \phi_1(xZ(G))\psi_1(xZ(G))
$$

and for every $y \in Z(G)$, $y^\gamma = \phi_0(y)\psi_0(y)$. But

$$
\phi_1(xZ(G))\psi_1(x) = (xZ(G))^{\gamma_\phi}(xZ(G))^{\gamma_\psi}
$$

and $\phi_0(y)\psi_0(y) = y^{\gamma_\phi}y^{\gamma_\psi}$. Thus $\gamma = \gamma_\phi + \gamma_\psi$, hence η is an additive homomorphism. On the other hand identities $\phi_1 \circ \psi_1(xZ(G)) = (xZ(G))^{\gamma_\psi\gamma_\phi} = (xZ(G))^{\gamma_\phi\gamma_\psi}$ and $\phi_0 \circ \psi_0(y) = y^{\gamma_\psi\gamma_\phi} = y^{\gamma_\phi\gamma_\psi}$ imply that η is a multiplicative homomorphism. The proposition is proved. $\square$

Proposition 5.3. *Let $\varphi : G \to H$ be an isomorphism of $QN_{2,n}$ groups. Then $P(f_G) \cong P(f_H)$.*

Proof. The isomorphism φ induces an isomorphism $\varphi_1 : G/Z(G) \to H/Z(H)$ and restricts to an isomorphism $\varphi_0 : Z(G) \to Z(H)$. We claim that if $(\phi_1, \phi_0) \in P(f_G)$ then

$$
(\varphi_1\phi_1\varphi_1^{-1}, \varphi_0\phi_0\varphi_0^{-1}) \in P(f_H)
$$

and

$$
\theta : P(f_G) \to P(f_H), \quad (\phi_1, \phi_0) \mapsto (\varphi_1\phi_1\varphi_1^{-1}, \varphi_0\phi_0\varphi_0^{-1}),
$$

is an isomorphism of rings.

Let $(\phi_1, \phi_0) \in P(f_G)$, $x, y \in G$, $\varphi(x) = z$ and $\varphi(y) = t$. We note that

$$
\begin{aligned}
f_H(\varphi_1(xZ(G)), \varphi_1(yZ(G))) &= f_H(zZ(H), tZ(H)) \\
&= [z, t] = [\varphi(x), \varphi(y)] \\
&= \varphi([x, y]) = \varphi_0([x, y]) \\
&= \varphi_0(f_G(xZ(G), yZ(G)))
\end{aligned}
$$

Next using this equality we get

$$
\begin{aligned}
f_H(\varphi_1 \phi_1 \varphi_1^{-1}(zZ(H)), tZ(H)) &= f_H(\varphi_1 \phi_1(xZ(G)), \varphi_1(yZ(G))) \\
&= \varphi_0(f_G(\phi_1(xZ(G), y(Z(G)))) \\
&= \varphi_0 \phi_0([x, y]) \\
&= \varphi_0 \phi_0 \varphi_0^{-1}([z, t]).
\end{aligned}
$$

With a similar argument we also get that

$$
f_H(zZ(H), \varphi_1 \phi_1 \varphi_1^{-1}(tZ(H))) = \varphi_0 \phi_0 \varphi_0^{-1}([z, t])
$$

which proves that $(\varphi_1 \phi_1 \varphi_1^{-1}, \varphi_0 \phi_0 \varphi_0^{-1}) \in P(f_H)$.

To prove that θ is surjective we should follow an argument similar to the above. Injectivity of θ as well as it being a ring homomorphism is clear. $\qquad\square$

Theorem 5.4. *Let R and S be commutative associative rings with unit. If*

$$
N_{2,n}(R, f_1, \ldots f_n) \cong N_{2,n}(S, q_1, \ldots q_n)
$$

as groups then $R \cong S$ as rings.

Proof. This is a direct corollary of Lemma 5.2 and Proposition 5.3. $\qquad\square$

6 Characterization of $Q_{2,n}$ groups over commutative associative rings

6.1 groups with a basis

Definition 6.1 (Basis)**.** *Let H be a group with distinct nontrivial elements h_1, $h_2, \ldots, h_n$. Let H_1, H_2, \ldots, H_n and H_{12}, \ldots, $H_{n-1,n}$ be subgroups of H satisfying the following conditions:*

1. *$H_i = C_H(h_i)$ and $H_{ij} = [h_i, H_j] = [H_i, h_j] = [H_i, H_j]$, $1 \leq i < j \leq n$,*

2. *$H_i \cap H_j = Z(H)$, $1 \leq i < j \leq n$,*

3. *$[H_i, H_i] = 1$, $1 \leq i \leq n$,*

4. *$[H, H] \subseteq Z(H)$,*

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5. (a) *every element of H can be written as $u_n u_{n-1} \dots u_1 v$ where each $u_i \in H_i$ and $v \in Z(H)$ and each u_i is unique modulo the center,*

   (b) *each $v \in Z(H)$ can be uniquely written as $u_{12} \dots u_{1n} u_{23} \dots u_{n-1,n}$ when $u_{ij} \in H_{ij}$.*

Then $\mathfrak{b} = \{h_1, h_2, \dots, h_n\}$ is called a basis for H.

Definition 6.2 ($P(f_H)$-basis)**.** *Let H be a group with a set of elements $\mathfrak{b} = \{h_1, \dots, h_n\}$. We call $\mathfrak{b}$ a $P(f_H)$-basis for H if*

- *$\mathfrak{b}$ is a basis for H,*

- *$H_i/Z(H)$ is a cyclic $P(f_H)$-module generated by $h_i Z(H)$, for all $1 \le i \le n$.*

Corollary 6.3. *Each $H_i/Z(H)$, $1 \le i \le n$, is a torsion free $P(f_H)$-module. Moreover each H_{ij} is also a torsion free $P(f_H)$ module generated by $[h_i, h_j]$.*

Proof. Pick $1 \le i \le n$ and suppose that there is an element $\alpha \in P(f_H)$ such that $(xZ(H))\alpha = 0$ for all $x \in H_i$. Pick $x \in H_i$ such that $x \notin Z(H)$. Let $i \ne j$, $1 \le j \le n$ and pick any $y \in H_j$ where $y \notin Z(H)$. Then

$$1 = f_H((xZ(H))^\alpha, yZ(H)) = f_H(xZ(H), (yZ(H))^\alpha) = [x, y'],$$

where $y' \in H_j$ and $y'Z(H) = (yZ(H))^\alpha$. So $y' \in H_i \cap H_j = Z(H)$. Since j was arbitrary by condition $5-a$ of the definition of a basis $\alpha \in Ann_{P(f_H)}(H/Z(H)) = 0$. The first statement follows. The second statement should also be clear now. $\square$

Lemma 6.4. *Let H be a group with elements $h_1, \dots, h_n$ constituting a basis for H. Then the ring $P(f_H)$ and its action on $H/Z(H)$ and $Z(H)$ are absolutely interpretable in H.*

Proof. The bilinear map f_H has width $\frac{n(n-1)}{2}$. Moreover the set $\{h_1, \dots h_n\}$ is a finite complete system for f_H. So by Theorem 2.3 the structure

$$\mathfrak{U}_{P(f_H)}(f_H) = \langle P(f_H), H/Z(H), Z(H), s_{H/Z(H)}, s_{Z(H)}, \delta_{f_H} \rangle,$$

where $s_{H/Z(H)}$ and $s_{Z(H)}$ describing the action of $P(f_H)$ on $H/Z(H)$ and $Z(H)$ respectively is absolutely interpretable in

$$\mathfrak{U}(f_H) = \langle H/Z(H), Z(H), \delta_{f_H} \rangle.$$

The factor group $H/Z(H)$ is absolutely interpretable in H. The subgroup $Z(H)$ is clearly definable without parameters. There is a formula of signature of groups describing the bilinear mapping f_H, since f_H is defined just by commutators, $Z(H)$ is absolutely definable in H and $H/Z(H)$ is absolutely interpretable in H. So $\mathfrak{U}(f_H)$ is absolutely interpretable in H. Therefore $\mathfrak{U}_{P(f_H)}(f)$ is absolutely interpretable in H. $\square$

Lemma 6.5. *Let G be a $QN_{2,n}$ group over R with a $P(f_G)$-basis*

$$\mathfrak{b} = \{g_1, \ldots, g_n\}.$$

If $\varphi : G \to H$ be an isomorphism of groups then the image $\mathfrak{b}^\varphi$ of $\mathfrak{b}$ under φ is a $P(f_H)$-basis for H.

Proof. Let $\varphi(g_i) = h_i$ for each $1 \leq i \leq n$. It is clear that $\mathfrak{b}^\varphi$ is a basis. It remains to prove that each $C_H(h_i)/Z(H) = (h_i Z(H))^{P(f_H)}$, $1 \leq i \leq n$. Let $\phi \in End(G/Z(G))$ and $\varphi_1 : G/Z(G) \to H/Z(H)$ be the isomorphism induced by φ. Let $xZ(G) = \phi(g_i Z(G))$, $x \in C_G(g_i)$ and $y = \varphi(x)$. Then

$$\begin{aligned}
\varphi_1 \phi \varphi_1^{-1}(h_i Z(H)) &= \varphi_1 \phi(g_i Z(G)) \\
&= \varphi_1(xZ(G)) \\
&= yZ(G).
\end{aligned}$$

Having this and comparing it with the isomorphism

$$\theta : P(f_G) \to P(f_H)$$

defined in the proof of Proposition 5.3 the result is clear. □

6.2   Characterization theorem

Theorem 6.6 (Characterization theorem). *Let $h_1, \ldots, h_n$ be some elements of H. Then the following are equivalent:*
(1) The group H has a $P(f_H)$-basis, $\mathfrak{b} = \{h_1, \ldots, h_n\}$;
(2) There is a commutative associative ring R and symmetric 2-cocycles

$$f^i : R^+ \times R^+ \to \oplus_{i=1}^{\binom{n}{2}} R^+,$$

such that
$$(H, \mathfrak{b}) \cong N_{2,n}^*(R, f^1, \ldots, f^n) = (G, \mathfrak{b}')$$

via an isomorphism $\varphi : (G, \mathfrak{b}') \to (H, \mathfrak{b})$ of enriched groups where $\mathfrak{b}' = \{g_1, \ldots, g_n\}$ is the standard basis for G.

If (1) holds each symmetric cocycle $f^i : R^+ \times R^+ \to \oplus_{i=1}^{\binom{n}{2}} R^+$, $1 \leq i \leq n$ is constructed such that each $H_i = C_H(h_i)$ is an abelian extension of $\oplus_{i=1}^{\binom{n}{2}} R^+$ by R^+ via the symmetric 2-cocycle f^i.

Proof. $(2) \Rightarrow (1)$ By Proposition 4.3 $\mathfrak{b}'$ is a basis of G. Since $\mathfrak{b}'$ is the standard basis of G it is clear that each $C_G(g_i)/Z(G)$ is a cyclic R-module. By proof of Proposition 5.2, for each element α of R there is a unique element of $P(f_G)$ acting on $G/Z(G)$ by α. So each $C_G(g_i)$ is a cyclic $P(f_G)$ module. So $\mathfrak{b}'$ is a $P(f_G)$ basis for G. Now Lemma 6.5 implies that $\mathfrak{b}$ is a $P(f_H)$-basis for H.
To prove $(1) \Rightarrow (2)$ we prove that the relations (a)-(d) of Lemma 4.4 hold with

a suitable choice for h_i^α among the representatives $(h_i Z(H))^\alpha$ for $\alpha \in R$ and $1 \le i \le n$. Consider the following maps:

$$\tau_{i,ij} : H_i \to H_{ij}, \quad x \mapsto [x, h_j],$$

and

$$\tau_{j,ij} : H_j \to H_{ij}, \quad x \mapsto [h_i, x].$$

for each $1 \le i < j \le n$. All the $\tau_{i,ij}$ and $\tau_{j,ij}$ are group homomorphisms by condition (4) of Definition 6.1 of a basis. They are surjective by condition (1) and they all have the same kernel, $Z(H)$ by (2) of definition of the basis. So $\tau_{i,ij}$ induces an isomorphism between $H_i/Z(H)$ and H_{ij} and $\tau_{j,ij}$ induces an isomorphism between $H_j/Z(H)$ and H_{ij}. So as each $H_i/Z(H)$ is a cyclic $P(f_H)$-module generated by $h_i Z(H)$, the element h_{ij} generates H_{ij} as a cyclic $P(f_H)$-module. Set $R = P(f_H)$. Then by Corollary 6.3,

$$\mu_{ij} : h_{ij}^R \to R^+, \quad h_{ij}^\zeta \mapsto \zeta,$$

is a group isomorphism. Moreover (5)-(b) of the definition of a basis together with above explanations implies that there is an isomorphism $\eta : \oplus_{i=1}^{\binom{n}{2}} R^+ \to Z(H)$. Now consider the following sequences of abelian groups:

$$0 \to \oplus_{i=1}^{\binom{n}{2}} R^+ \xrightarrow{\eta} H_i \xrightarrow{\mu_{i,i+1}\tau_{i,ii+1}} R^+ \to 0,$$

where $1 \le i \le n-1$ and

$$0 \to \oplus_{i=1}^{\binom{n}{2}} R^+ \xrightarrow{\eta} H_n \xrightarrow{\mu_{n-1,n}\tau_{n,(n-1,n)}} R^+ \to 0.$$

They are clearly exact. Let for each $1 \le i \le n$, $f^i : R^+ \times R^+ \to \oplus_{i=1}^{\binom{n}{2}} R^+$ be the 2-cocycle corresponding to the extension above. Each f^i is clearly a symmetric 2-cocycle since H_i is abelian by condition (3) of Definition 6.1, hence H_i is an abelian extension of $\oplus_{i=1}^{\binom{n}{2}} R^+$ by R^+ via the 2-cocycle f^i. Therefore $H_i \cong R^+ \times \oplus_{i=1}^{\binom{n}{2}} R^+ = K_i$, $1 \le i \le n$, as groups when the multiplication:

$$\begin{aligned}
xy &= (\alpha, \gamma_{12}, \dots, \gamma_{n-1,n})(\alpha', \gamma'_{12}, \dots, \gamma'_{n-1,n}) \\
&= (\alpha + \alpha', \gamma_{12} + \gamma'_{12} + f^i_{12}(\alpha, \alpha'), \dots, \\
&\quad \gamma_{n-1,n} + \gamma'_{n-1,n} + f^i_{n-1,n}(\alpha, \alpha')),
\end{aligned}$$

is assumed on K_i and $f^i = (f^i_{12}, \dots, f^i_{n-1,n})$. Suppose $\eta_i : K_i \to H_i$ be the group isomorphism whose existence established above. Now for each $1 \le i \le n$ and $\alpha \in R$ let $h_i^\alpha \in H_i$ be the element of the equivalence class $(h_i Z(H))^\alpha$ such that

$$h_i^\alpha = \eta_i(\alpha, \underbrace{0, \dots, 0}_{\binom{n}{2}-\text{times}}).$$

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Firstly notice that $h_i^\alpha = 1$ if and only if $\alpha = 0$. Moreover it is clear that for each $1 \le i \le n$ and $\alpha, \beta \in R$:

$$h_i^\alpha h_i^\beta = h_i^{\alpha+\beta} h_{12}^{f_{12}^i(\alpha,\beta)} \ldots h_{n-1,n}^{f_{n-1,n}^i(\alpha,\beta)}.$$

Thus the relations (c) of Lemma 4.4 hold between h_i^α, $1 \le i \le n$ and $\alpha \in R$. We also note that,

$$\begin{aligned}
[h_i^\alpha, h_j^\beta] &= f_H((h_i Z(H))^\alpha, (h_j Z(H))^\beta) \\
&= (f_H((h_i Z(H)), (h_j Z(H))))^{\alpha\beta} \\
&= [h_i, h_j]^{\alpha\beta} \\
&= h_{ij}^{\alpha\beta}
\end{aligned}$$

for $1 \le i < j \le n$, and $\alpha, \beta \in R$, which proves that relations (a) hold. Relations (b) hold since each h_{ij}^α, $1 \le i < j \le n$ and $\alpha \in R$, is central. Relations (d) hold also in H by the fact that each H_{ij} is an R-module.

The set,

$$\mathcal{H} = \{h_i^\alpha, h_{kl}^\beta : 1 \le i \le n, 1 \le k < l \le n, \quad \alpha, \beta \in R\},$$

generates H as a group by (5) of the definition of a basis. Let F be the free group on $\mathcal{H}$. Let $\mathcal{R}$ be the normal closure of the relations in the lemma 4.4 in F, g_i and g_{ij} substituted by h_i and h_{ij} and the exponents come from the ring R defined here and $\odot$ taken to be concatenation. Let $\langle \mathcal{H} | \mathcal{R} \rangle$ be F modulo the normal subgroup $\mathcal{R}$. Consider the mapping:

$$\langle \mathcal{H} | \mathcal{R} \rangle \longrightarrow H, \quad h_i^\alpha \mapsto h_i^\alpha, \quad h_{kl}^\alpha \mapsto h_{kl}^\alpha$$

for $\alpha \in R$, $1 \le i \le n$ and $1 \le k < l \le n$. The map is a well defined homomorphism of groups since as proved above the relations (a)-(d) of 4.4 hold also in H. The map is also surjective since $\mathcal{H}$ generates H. Every word W in $\langle \mathcal{H} | \mathcal{R} \rangle$ is equivalent to a word of the form $h_n^{\alpha_n} \ldots h_1^{\alpha_1} h_{12}^{\alpha_{12}} \ldots h_{n-1,n}^{\alpha_{n-1,n}}$. So W maps to an element h of H of this form. The uniqueness of this form for h in H is guarantied by (5) of 6.1. So if h is trivial in H then all the exponents in the above form are zero so the word W is trivial in $\langle \mathcal{H} | \mathcal{R} \rangle$. So $\langle \mathcal{H} | \mathcal{R} \rangle \cong H$. But by Lemma 4.4, $\langle \mathcal{H} | \mathcal{R} \rangle \cong N_{2,n}(R, f^1, \ldots f^n)$, hence

$$N_{2,n}^*(R, f^1, \ldots, f^n) \cong (H, \mathfrak{b}).$$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad \square$

Lemma 6.7. *Let $\mathfrak{b} = \{h_1, \ldots, h_n\}$ be a basis for a group H. Then the subgroups $Z(H)$, H_i, $1 \le i \le n$, H_{ij}, $1 \le i < j \le n$, and $[H, H]$ are first order definable in the enriched group $(H, \mathfrak{b})$. Thus all the conditions (1)-(5) of the definition of basis can be expressed by first order formulas of an enriched signature of groups. Moreover the additional condition making a basis $\mathfrak{b}$ a $P(f_H)$ basis is also expressible in the signature of groups.*

Proof. The center is defined by the formula

$$\phi_{Z(H)}(x) : \forall y [x, y] = 1.$$

Let $\bar{h} = (h_1, \ldots, h_n)$. Then For each $1 \le i \le n$, H_i is defined by:

$$\phi_{H_i}(\bar{h}, x) : [h_i, x] = 1.$$

For each $1 \le i < j \le n$, the subgroup H_{ij} is generated by the set $\{[h_i, y] : y \in H_j\}$. So for every element x of H_{ij}, for some fixed $1 \le i < j \le n$, can be written as a product

$$x = [h_i, y_1] \ldots [h_i, y_m], \quad y_1, \ldots, y_n \in H_j.$$

Since H is a 2-nilpotent group, by condition (4) of the definition of a basis, we can rewrite x as $x = [h_i, y_1 \ldots y_n]$. So the subgroup H_{ij} is defined by the formula:

$$\phi_{H_{ij}}(\bar{h}, x) : \exists y (x = [h_i, y] \wedge \phi_{H_j}(y)).$$

By (4), $[H, H]$ sits inside the center. Therefore every element x of $[H, H]$ has the form mentioned in (5)-(b). Conversely if some arbitrary element $x \in H$ has the form indicated in (5)-(b), then $x \in [H, H]$ since $H_{ij} \subseteq [H, H]$ for each $1 \le i < j \le n$. Thus $[H, H]$ is defined in $(H, \mathcal{B})$ by the formula:

$$\exists y_{12} \ldots y_{n-1,n} \big(\bigwedge_{1 \le i < j \le n} \phi_{H_{ij}}(y_{ij}) \wedge x = y_{12} \ldots y_{n-1,n} \big).$$

It is now clear how to formulate conditions (1),(2),(3) and (5). Condition (4) is simply given by:

$$\forall x, y, z ([x, y].z = z.[x, y]).$$

Lemma 6.4shows clearly that there is a first order sentence ϕ of the signature of groups such that

$$H_i/Z(H) = (h_i Z(H))^{P(f_H)} \Leftrightarrow H \models \phi$$

for all $1 \le i \le n$. $\qquad\square$

6.3 Groups elementarily equivalent to a free 2-nilpotent group of arbitrary finite rank

Theorem 6.8. *Let R be a commutative associative ring with unit and $G = N_{2,n}(R, f_1, \ldots f_n)$. If a group H is elementarily equivalent to G then $H \cong N_{2,n}(S, q_1, \ldots q_n)$ for some ring S such that $R \equiv S$.*

Proof. The standard basis of G is a $P(f_G)$-basis (see the $(2) \Rightarrow (1)$ of the proof of the characterization theorem). Since $G \equiv H$ the formulas that interpret f_H in H are the same as the ones which interpret f_G in G. Moreover f_H has the same width as f_G. So the formulas that interpret the action of $P(f_G)$ on $G/Z(G)$ in G are the same as the ones that interpret action of $P(f)H$ on $H/Z(H)$ in H. So $P(f_G) \equiv P(f_H)$ and moreover H has to have a $P(f_H)$-basis. So by the characterization theorem $H \cong N_{2,n}(S, q^1, \ldots, q^n)$ for some ring $S = P(f_H) \equiv P(f_G) \cong R$ and the relevant symmetric 2-cocycles q^i. $\qquad\square$

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The main result of this paper is just a corollary of Theorem 6.8

Theorem 6.9. *Let G be a free 2-nilpotent group of rank n. If H is a group elementarily equivalent to G then H has the form $N_{2,n}(R, f_1, \ldots f_n)$ for some ring $R \equiv \mathbb{Z}$.*

Proof. By Proposition 3.2, $G \cong N_{2,n}(\mathbb{Z})$. By Theorem 6.8, H has the form indicated in the statement of the theorem. □

7 Central extensions and elementary equivalence

The aim of this section is to prove that for any two elementarily equivalent characteristic zero integral domains R and S

$$N_{2,n}(R, f^1, \ldots, f^n) \equiv N_{2,n}(S, g^1, \ldots, g^n)$$

for any symmetric 2-cocycles $f^1, \ldots f^n, g^1, \ldots g^n$ described before. Meanwhile we prove that any two central extensions of a torsion free abelian group A by a torsion free abelian group B are elementarily equivalent providing that the 2-cocycles corresponding to the extensions differ up to a 2-coboundary by a symmetric 2-cocycle.

Lemma 7.1. *Let*

$$0 \to A \to G \to B \to 0$$

be a central extension of the abelian groups A and B. Let $(I, \mathcal{D})$ be an ultrafilter. Then $G^I/\mathcal{D}$ is isomorphic to a central extension of $A^I/\mathcal{D}$ by $B^I/\mathcal{D}$.

Proof. Let f be the 2-cocycle corresponding to the extension above. Let $[x]$ denote an element of $B^I/\mathcal{D}$ when $x \in B^I$. We define a 2-cocycle:

$$f^{\mathcal{D}} : B^I/\mathcal{D} \times B^I/\mathcal{D} \to A^I/\mathcal{D}, \quad ([x], [y]) \mapsto [f^I(x, y)]$$

where $f^I : B^I \times B^I \to A^I$ is the 2-cocycle defined by $f^I(x, y)(i) = f(x(i), y(i))$, $i \in I$. To prove the well-definedness let $[x] = [z], [y] = [t] \in B^I/\mathcal{D}$. Then, $A = \{i \in I : x(i) = z(i)\} \in \mathcal{D}$ and $B = \{i \in I : y(i) = t(i)\} \in \mathcal{D}$. If $C = A \cap B$ then $C \in \mathcal{D}$ and

$$C \subseteq \{i \in I : f(x(i), y(i)) = f(z(i), t(i))\} = \{i \in I : f^I(x, y)(i) = f^I(z, t)(i)\},$$

which implies that $f^{\mathcal{D}}([x], [y]) = f^{\mathcal{D}}([z], [t])$.

Let H be the central extension of $A^I/\mathcal{D}$ by $B^I/\mathcal{D}$ induced by $f^{\mathcal{D}}$. we denote an element of $G^I/\mathcal{D}$ by $[(b, a)]$ and an element of H by $([b], [a])$ where $b \in B^I$ and $a \in A^I$. Define:

$$\phi : G^I/\mathcal{D} \to H, \quad [(b, a)] \mapsto ([b], [a])$$

We prove that ϕ is an isomorphism of groups. To prove well-definedness let $[(b, a)] = [(d, c)]$ for $b, d \in B^I$ and $a, c \in A^I$. Let

$$C = \{i \in I : (b(i), a(i)) = (d(i), c(i))\}.$$

Then $C \subseteq D, E$ where $D = \{i \in I : b(i) = d(i)\}$ and $E = \{i \in I : a(i) = c(i)\}$. So $D, E \in \mathcal{D}$ since C is. Hence $([b], [a]) = ([d], [c])$. Surjectivity of ϕ is clear. To prove injectivity assume that $([b], [a]) = (0, 0)$ for $b \in B^I$ and $a \in A^I$. Let $D' = \{i \in I : b(i) = 0\}$ and $E' = \{i \in I : a(i) = 0\}$. So $D', E' \in \mathcal{D}$. But

$$\{i \in I : (b(i), a(i)) = (0, 0)\} = D' \cap E' \in \mathcal{D},$$

which implies that $[(b, a)] = [(0, 0)]$. It can be easily checked that ϕ is a homomorphism. □

Let A and B are two abelian groups and $f \in Z^2(B, A)$. By (A, B, f) we denote the central extension of A by B corresponding to f.

Lemma 7.2. *Let A and B be torsion free abelian groups and $G = (A, B, f)$ and $H = (A, B, g)$ be as above. If $f - g \in S^2(B, A)$ then $G \equiv H$.*

Proof. Let $(I, \mathcal{D})$ be an ultrafilter over which $A^I/\mathcal{D}$ is ω_1-saturated. By Lemma 7.1 there are $f^{\mathcal{D}}, g^{\mathcal{D}} \in H^2(B^I/\mathcal{D}, A^I/\mathcal{D})$ such that $G^I/\mathcal{D} \cong (A^I/\mathcal{D}, B^I/\mathcal{D}, f^{\mathcal{D}})$ and $H^I/\mathcal{D} \cong (A^I/\mathcal{D}, B^I/\mathcal{D}, g^{\mathcal{D}})$. By Theorem 1.11 of [13], $A^I/\mathcal{D}$ is pure injective. Moreover $B^I/\mathcal{D}$ is torsion free since B is. These facts imply that

$$\text{Ext}(B^I/\mathcal{D}, A^I/\mathcal{D}) = 0.$$

Note also that $f^{\mathcal{D}} - g^{\mathcal{D}} \in S^2(B^I/\mathcal{D}, A^I/\mathcal{D})$. Hence $f^{\mathcal{D}}$ and $g^{\mathcal{D}}$ are cohomologous. So

$$G^I/\mathcal{D} \cong (A^I/\mathcal{D}, B^I/\mathcal{D}, f^{\mathcal{D}}) \cong (A^I/\mathcal{D}, B^I/\mathcal{D}, g^{\mathcal{D}}) \cong H^I/\mathcal{D}.$$

Hence $G \equiv H$. □

Corollary 7.3. *Let R be a characteristic zero integral domain and consider $N_{2,n}(R, f^1, \ldots, f^n)$ for some symmetric 2-cocycles f^i. Then*

$$N_{2,n}(R, f^1, \ldots, f^n) \equiv N_{2,n}(R).$$

Proof. Let

$$(\bar{\alpha}, \bar{\beta}) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_n) \in R^n \times R^n,$$

and for each $1 \le i \le n$ define

$$q^i(\bar{\alpha}, \bar{\beta}) =_{df} f^i(\alpha_i, \beta_i), \quad \forall (\bar{\alpha}, \bar{\beta}) \in R^n \times R^n.$$

We note that $N_{2,n}(R, f^1, \ldots, f^n)$ is defined by the 2-cocycle

$$f = (f_{ij}) : \oplus_{i=1}^n R^+ \times \oplus_{i=1}^n R^+ \to \oplus_{i=1}^{\binom{n}{2}} R^+,$$

Where

$$f_{ij}(\bar{\alpha}, \bar{\beta}) = \alpha_i \beta_j + \sum_{i=1}^n q^i(\bar{\alpha}, \bar{\beta}), \quad \forall (\bar{\alpha}, \bar{\beta}) \in R^n \times R^n.$$

Now to conclude we need to observe that $\sum_{i=1}^n q^i \in S^2(\oplus_{i=1}^n R^+, \oplus_{i=1}^{\binom{n}{2}} R^+)$ and use Lemma 7.1. □

Lemma 7.4. *If $R \equiv S$ as rings then $N_{2,n}(R) \equiv N_{2,n}(S)$.*

Proof. We just need to make the simple observation that the group $N_{2,n}(R)$ is interpretable in the ring R with the same formulas that interpret $N_{2,n}(S)$ in S. $\qquad\square$

The following statement is the obvious corollary of Lemma 7.3 and Lemma 7.4.

Corollary 7.5. *If R and S are elementarily equivalent characteristic zero integral domains then*

$$N_{2,n}(R, f^1, \ldots, f^n) \equiv N_{2,n}(S, g^1, \ldots, g^n),$$

For any $f^i \in S^2(\oplus_{i=1}^n R^+, \oplus_{i=1}^{\binom{n}{2}} R^+)$ and $g^i \in S^2(\oplus_{i=1}^n S^+, \oplus_{i=1}^{\binom{n}{2}} S^+)$.

8 A group elementarily equivalent to $N_{2,n}(\mathbb{Z})$ which is not $N_{2,n}$

In this section we prove the existence of a $QN_{2,n}$-group over a certain ring which is not a $N_{2,n}$-group over any commutative associative ring.

Lemma 8.1. *assume that $\varphi : G = N_{2,n}(R, f^1, \ldots, f^n) \to N_{2,n}(S) = H$ is an isomorphism of groups. If $\varphi_1 : Ab(G) \to Ab(H)$ is the isomorphism induced by φ and $\varphi_0 : Z(G) \to Z(H)$ is the restriction of φ to $Z(G)$ then there exists an isomorphism $\mu : R \to S$ such that*

$$\varphi_1((xZ(G))^\alpha) = (\varphi(x)Z(H))^{\mu(\alpha)}, \quad \text{for all } x \in G \text{ and } \alpha \in R,$$

and

$$\varphi_0(x^\alpha) = (\varphi(x))^{\mu(\alpha)}, \quad \text{for all } x \in Z(G) \text{ and } \alpha \in R.$$

Proof. Let $\theta : P(f_G) \to P(f_H)$ be the isomorphism obtained in Proposition 5.3. By Proposition 5.2 the map:

$$\mu : R \to S, \quad \alpha_\phi \mapsto \alpha_{\theta(\phi)}$$

is an isomorphism of rings. Then

$$
\begin{aligned}
\varphi_1((xZ(G))^{\alpha_\phi}) &= \varphi_1 \phi_1(xZ(G)) \\
&= \varphi_1 \phi_1 (\varphi_1)^{-1} \varphi_1(xZ(G)) \\
&= \theta(\phi_1)\varphi_1(xZ(G)) \\
&= (\varphi_1(xZ(G)))^{\mu(\alpha_\phi)} \\
&= (\varphi(x)Z(H))^{\mu(\alpha_\phi)}
\end{aligned}
$$

for all x in G. A similar argument using ϕ_0 instead of ϕ_1 proves that

$$\varphi(x^\alpha) = \varphi(x)^{\mu(\alpha)}, \quad \text{for all } x \in Z(G) \text{ and } \alpha \in R.$$

$\qquad\square$

Theorem 8.2. *Let R be a binomial domain and*

$$\varphi : G = N_{2,n}(R, f^1, \ldots, f^n) \to N_{2,n}(S) = H$$

be an isomorphism of groups. Then for each $1 \le j \le r$ we have that $f^j \in Z^2(R^+, \oplus_{i=1}^{\binom{n}{2}} R^+)$, i.e. each f^j is a 2-coboundary.

Proof. Let $\mathfrak{b} = \{g_1, \ldots, g_n\}$ be a standard basis for G and the g_{ij} be defined as usual. Set $\varphi(g_i) = h_i$ and $\varphi(g_{ij}) = h_{ij}$. Let $\varphi_1 : Ab(G) \to Ab(H)$ be the group isomorphism induced by φ. By Lemma 8.1 there exists an isomorphism $\mu : R \to S$ of rings so that $\varphi_1((xZ(G))^\alpha) = (\varphi_1(x\Gamma_2(G)))^{\mu(\alpha)}$, for all α in R and x in G. This implies that $\{h_1 Z(H), \ldots, h_n Z(H)\}$ generates $H/\Gamma_2(H)$ freely as an S-module since $\{g_1 Z(G), \ldots, g_n Z(G)\}$ generates $G/\Gamma_2(G)$ freely as an R-module. So $\mathfrak{c} = \{h_1, \ldots, h_n\}$ generates H as an S-group (see Remark 1 and [10], 4.1). So every element h of H has a unique representation

$$h = h_n^{\alpha_n} \cdots h_1^{\alpha_1} h_{12}^{\gamma_{12}} \cdots h_{n-1,n}^{\gamma_{n-1,n}},$$

where the α_i and γ_{ij} are elements of S. For simplicity let us denote by $\mathbf{h}_2^\gamma$ the product $h_{12}^{\gamma_{12}} \cdots h_{n-1,n}^{\gamma_{n-1,n}}$. By Lemma 8.1

$$\phi(g_i^\alpha) = h_i^{\mu(\alpha)} \mathbf{h}_2^{g(\mu(\alpha))}, \quad \forall \alpha \in R,$$

where $g = (g_{ij}) : S \to \prod_{i=1}^{\binom{n}{2}} S$ is a function determined by φ. Choose two arbitrary elements β and β' in S. Then,

$$
\begin{aligned}
h_i^{\beta+\beta'} &= h_i^\beta h_i^{\beta'} \\
&= \varphi(g_i^{\mu^{-1}(\beta)}) \mathbf{h}_2^{-g(\beta)} \varphi((g_i)^{\mu^{-1}(\beta')}) \mathbf{h}_2^{-g(\beta')} \\
&= \varphi(g_i^{\mu^{-1}(\beta)}) \varphi(g_i^{\mu^{-1}(\beta')}) \mathbf{h}_2^{-g(\beta)-g(\beta')} \\
&= \varphi(g_i^{\mu^{-1}(\beta+\beta')} \mathbf{h}_2^{f^i(\mu^{-1}(\beta), \mu^{-1}(\beta'))}) \mathbf{h}_2^{-g(\beta)-g(\beta')} \\
&= h_i^{\beta+\beta'} \mathbf{h}_2^{\mu f^i(\mu^{-1}(\beta), \mu^{-1}(\beta')) + g(\beta+\beta') - g(\beta) - g(\beta')}
\end{aligned}
$$

The identity above clearly shows that the $\mu f^i(\mu^{-1}(-), \mu^{-1}(-)) \in Z^2(S^+, \oplus_{i=1}^{n_c} S^+)$. Since μ is a ring isomorphism this implies that for each $1 \le j \le r$, f^i is a 2-coboundary as claimed. $\qquad\square$

Lemma 8.3 (Belegradeck)**.** *There is a ring R, $R \equiv \mathbb{Z}$ such that $Ext(R^+, R^+) \neq 0$.*

Proof. See [7]. $\qquad\square$

Theorem 8.4. *Let $G \cong N_{2,n}(\mathbb{Z})$, i.e. G be a free nilpotent group of rank n and class 2. There is a $QN_{2,n}$ group H such that $G \equiv H$ but H is not an $N_{2,n}$ group over any commutative associative ring.*

Proof. Let R be ring $R \equiv \mathbb{Z}$ such that $Ext(R^+, R^+) \neq 0$. By Corollary 8.2 there are 2-cocycles $f^i : R^+ \times R^+ \to \oplus_{i=1}^{\binom{n}{2}} R^+$, $1 \leq i \leq n$, such that

$$H = N_{2,n}(R, f^1, \ldots, f^n) \ncong N_{2,n}(R).$$

We note that $H \ncong N_{2,n}(S)$ for any associative commutative ring S by Theorem 5.4. Moreover $H \equiv G$ by Lemma 7.5. $\square$

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